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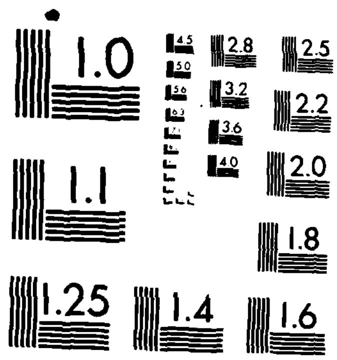

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CENTER FOR STOCHASTIC PROCESSES

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON STOCHASTIC INTEGRATION BY SERIES OF WIENER INTEGRALS

by

Jan Rosinski

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ON STOCHASTIC INTEGRATION BY SERIES OF WIENER INTEGRALS

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Abstract

Stochastic integrals of random functions with respect to a white noise random measure are defined in terms of random series of usual Wiener integrals. Conditions for the existence of such integrals are obtained in terms of the nuclearity of certain operators on L^2-spaces. The relation with the Fisk-Stratonovich symmetric integral is also discussed.

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1. Introduction

Before introducing and discussing our results, let us introduce some notation and definitions which are used throughout the paper. $(T,\mathcal{A},m)$ denotes an atomless separable σ-finite measure space and $(\Omega,\mathcal{F},P)$ a probability space. $W = \{W(A)\colon A \in \mathcal{A}_0\}$ is a white noise random measure on $\mathcal{A}_0 = \{A \in \mathcal{A}\colon m(A) < \infty\}$ with control measure m, i.e. W is a mean-zero Gaussian stochastic process indexed by sets from $\mathcal{A}_0$ and with covariance function $EW(A)W(B) = m(A\cap B)$. $\xi = \{\xi(t)\colon t \in T\}$ is a measurable real valued stochastic process such that

$$\int_T E|\xi(t)|^2 m(dt) < \infty.$$

Let $\{\phi_n\}$ be a CONS in (real) $L^2(T)$ and

$$a_n = a_n(\omega) = \int_T \xi(t,\omega)\phi_n(t)m(dt).$$

Clearly

$$\xi(t,\omega) = \sum_n a_n(\omega)\phi_n(t) \quad \text{in } L^2(T \times \Omega).$$

A stochastic integral $\int_T \xi * dW$ is defined by

$$(1.1) \qquad \int_T \xi * dW = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \int_T \phi_n(t)dW(t),$$

provided the limit exists in $L^2(\Omega)$ and does not depend on the choice of a CONS $\{\phi_n\}$.

This is a very attractive definition of a stochastic integral that does not require any special kind of measurability of ξ and the parameter set can be arbitrary.

A stochastic integral of this type has been defined and studied by Balkan [1], Kuo and Rusek [6] and Ogawa [8], independently. Ogawa [8] has proven that L^1-convergence in (1.1) with respect to the trigonometric basis $\{\phi_n\}$ implies similar convergence to the same limit with respect to Haar basis. Balkan [1] and Kuo and Rusek [6] studied the case when $\xi(t)$, $t \in T$ is a Wiener L^2-functional of W, i.e. for every $t \in T$, $\xi(t)$ is $F^W = \sigma\{W(A): A \in A_0\}$-measurable. Kuo and Rusek [6] (cf. also [7]) using Hida's white noise analysis studied sufficient conditions for the convergence in (1.1) and proved that under certain assumptions $\int_0^1 \xi * dW$ can be evaluated utilizing the Fisk-Stratonovich procedure.

Our approach is based on some classical results from the theory of nuclear operators on Hilbert spaces and on the Ito-Wiener expansion of L^2-Wiener functionals. In Section 2, we establish characterization of integral operators with a summable trace, which is basic for this paper. In Section 3, we study special cases of integrands. We show that if ξ is a Gaussian process subordinate to W, then $\int \xi * dW$ is a quadratic form in independent standard normal r.v.'s as it was studied by Varberg [10]. A necessary and sufficient condition for the existence of $\int \xi * dW$ when ξ lies in the p-th homogeneous chaos is given in Theorem 3.3. A general sufficient condition for the integrability of ξ is given in Theorem 4.1. In Section 5, we investigate the relationship between $\int_0^1 \xi * dW$ and the Fisk-Stratonovich integral. Theorem 5.8 provides sufficient condition for the existence and equality of both integrals. This condition, which is given in terms of appropriate Sobolev-space norms, is of the same nature as the one presented in [6], but differs in the value of a coefficient ((p + 1)! instead of p! in [6]). Theorem 5.9 gives a quite simple condition which guarantees the evaluation of $\int_0^1 \xi * dW$ as a limit of corresponding Stiltjes sums.

Throughout this paper the following notations are used:

$$L^2(T^k) := L^2(T^k, \overset{k}{\underset{j=1}{\otimes}} A_j, \overset{k}{\underset{j=1}{\otimes}} m_j),$$

where $A_j = A$, $m_j = m$;

$$I_p(g) = \underbrace{\int_T \cdots \int_T}_{p\text{-times}} g(s_1, \ldots, s_p)dW(s_1)\ldots dW(s_p)$$

is the p-tuple Itô-Wiener integral of $g \in L^2(T^p)$; when (T,m) is the unit interval with Lebesgue measure, then dW is replaced by dB, where $B(t)$, $t \in [0,1]$ is a standard Brownian motion; $\int_0^1 \xi(t)dB(t)$ is the usual Itô integral of a nonanticipating process $\xi(t)$, $t \in [0,1]$; $\int_0^1 \xi(t) \circ dB(t)$ denotes the Fisk-Stratonovich integral (cf. [4], p. 101).

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2. Integral operators with a summable trace

Let H be a real separable Hilbert space, and $k: T^2 \to H$ be a measurable mapping such that $\int_{T^2} \|k(s,t)\|^2 dm(s)dm(t) < \infty$. Define an operator $K: L^2(T) \to L^2(T;H)$ by

$$(2.1) \qquad (K\phi)(t) = \int_T \phi(s)k(s,t)dm(s), \quad \phi \in L^2(T).$$

We say that K *has a summable trace* if for every CONS $\{\phi_n\} \subset L^2(T)$ the series

$$(2.2) \qquad \sum_n \int_{T^2} \phi_n(s)k(s,t)\phi_n(t)dm(s)dm(t)$$

converges in H.

Let $\tilde{k}$ be the symmetrization of k, i.e.

$$\tilde{k}(s,t) = 2^{-1}\{k(s,t) + k(t,s)\}, \quad s,t \in T,$$

and let $\tilde{K}: L^2(T) \to L^2(T;H)$ be the corresponding integral operator with kernel $\tilde{k}$. Note that K has a summable trace if and only if $\tilde{K}$ possesses this property and the limit in (2.2) is the same if k is replaced by $\tilde{k}$.

<u>Proposition 2.1.</u> *An operator* K *given by* (2.1) *has a summable trace if and only if for every* $h \in H$ *the operator* $\tilde{K}_h: L^2(T) \to L^2(T)$ *defined by*

$$(\tilde{K}_h\phi)(t) = \int_T \phi(s)\langle \tilde{k}(s,t),h\rangle dm(s), \quad \phi \in L^2(T),$$

is nuclear.

Proof. Let $\{\phi_n\}$ be a CONS in $L^2(T)$ and put $h_n = \int_{T^2}\phi_n(s)k(s,t)\phi_n(t)dm(s)dm(t)$.

Assume that K has a summable trace. Since any permutation of $\{\phi_n\}$ is also a CONS in $L^2(T)$, $\Sigma_n h_n$ converges unconditionally in H. Hence for every $h \in H$

$$(2.3) \qquad \sum_n |\langle \tilde{K}_h\phi_n,\phi_n\rangle| = \sum_n |\langle h_n,h\rangle| < \infty.$$

Since $\tilde{K}_h$ is also a selfadjoint operator, $\tilde{K}_h$ is nuclear (cf. e.g. [2], Theorem 3.4.3).

Conversely, assume $\tilde{K}_h$ is a nuclear operator for every $h \in H$. Thus $\Sigma_n|\langle\tilde{K}_h\phi_n,\phi_n\rangle| < \infty$ for every CONS $\{\phi_n\}$ in $L^2(T)$, and by (2.3) $\Sigma_n h_n$ converges weakly unconditionally in H. Since H is weakly complete, $\Sigma_n h_n$ converges strongly in H; cf. e.g. [3], II.5.

Corollary 2.2. *If* K *has a summable trace, then* $h \to \mathrm{Trace}(\tilde{K}_h)$ *is a linear functional on* H *satisfying the equality*

$$\langle h_0,h\rangle = \mathrm{Trace}\ (\tilde{K}_h), \quad h \in H,$$

where h_0 *is the limit of the series (2.2) for some (any)* CONS $\{\phi_n\}$ *in* $L^2(T)$. *Hence the trace of* K, *denoted by* trk *and given by the series*

$$\mathrm{trk} := \sum_n \int_{T^2} \phi_n(s)k(s,t)\phi_n(t)dm(s)dm(t)$$

is well-defined, i.e. does not depend on the choice of a CONS $\{\phi_n\}$ in $L^2(T)$.

3. Integration in some special cases

A. Integrals of subordinate Gaussian processes.

Let

$$\xi(t) = \int_T f(s,t)dW(s) = I_1(f(\cdot,t)), \quad t \in T, \tag{3.1}$$

where f belongs to $L^2(T^2)$. Let $\{\phi_n\}$ be a CONS in $L^2(T)$ and consider the orthonormal expansion of f in $L^2(T^2)$:

$$f(s,t) = \sum_{m,n} f_{mn}\phi_m(s)\phi_n(t),$$

and the i.i.d. standard normal r.v.'s $X_n = \int_T \phi_n(s)dW(s)$.

Proposition 3.1. *Let* ξ *be given by (3.1). Then* $\int_T \xi{*}dW$ *exists and*

$$\int_T \xi{*}dW = \sum_{m,n} f_{mn}X_mX_n \qquad a.s.$$

provided the operator $F:L^2(T) \to L^2(T)$ *defined by*

$$(F\phi)(t) = \int_T \phi(s)f(s,t)dm(s), \quad \phi \in L^2(T),$$

has a summable trace.

Note that in this case $\int \xi{*}dW$ coincides with the double stochastic integral of f defined by Varberg [10], which is different from the double Itô-Wiener integral $I_2(f)$.

Proof. Since $\Sigma_n f_{nn} = \mathrm{tr} f$ (cf. Corollary 2.2), the series $\Sigma_n f_{nn}$ converges unconditionally. Therefore $\Sigma_{m,n} f_{mn}X_mX_n$ converges unconditionally in $L^2(\Omega)$ (cf. [10]), and

$$\sum_{m,n} f_{mn} X_m X_n = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\sum_{m=1}^{\infty} f_{mn} X_m \right) X_n$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\int_T \xi(t)\phi_n(t)\,dm(t) \right) X_n$$

$$= \int_T \xi(t) * dW(t). \qquad \square$$

Example 3.2. $\int_0^1 B*dB = B^2(1)/2$.

Proof. Indeed, $B(t) = \int_0^1 1_D(s,t)\,dB(s)$, where $D = \{(s,t): 0 \le s \le t \le 1\}$, and $\tilde{F}\phi = 2^{-1}\langle 1_{[0,1]}, \phi\rangle 1_{[0,1]}$ is nuclear as a one-dimensional projection, where $\tilde{F}$ is the symmetrization of F. Also

$$f_{mn} + f_{nm} = 2\langle \tilde{F}\phi_m, \phi_n\rangle = \langle 1_{[0,1]}, \phi_m\rangle\langle 1_{[0,1]}, \phi_n\rangle.$$

Therefore, by Proposition 3.1

$$\int_0^1 B*dB = 2^{-1} \lim_{N\to\infty} \sum_{m,n\le N} (f_{mn} + f_{nm}) X_m X_n$$

$$= 2^{-1} \lim_{N\to\infty} \left(\sum_{n=1}^{N} \langle 1_{[0,1]}, \phi_n\rangle X_n \right)^2 = 2^{-1} B^2(1). \qquad \square$$

B. Integrals of multiple Itô-Wiener integrals.

Let

(3.2) $$\xi(t) = \int_{T^p} f(s_1, \ldots, s_p, t)\,dW(s_1)\ldots dW(s_p) = I_p(f(\cdot,t)),$$

$t \in T$, where $f = f(s_1, \ldots, s_p, t)$ belongs to $L^2(T^{p+1})$ and is symmetric in $s_1, \ldots, s_p$ for each fixed t. We have for every $t \in T$

$$\|\xi(t)\|^2_{L^2(\Omega)} = p!\|f(\cdot,t)\|^2_{L^2(T^{p+1})}$$

and

$$\|\xi\|^2_{L^2(T\times\Omega)} = p!\|f\|^2_{L^2(T^{p+1})}.$$

Since for a.e. $(s,t) \in T^2$, $f(\cdot,s,t) \in L^2(T^{p-1})$, the mapping $F:L^2(T) \to L^2(T;L^2(T^{p-1}))$ given by

$$(3.3) \qquad (F\phi)(t) = \int_T \phi(s)f(\cdot,s,t)\,dm(s), \quad \phi \in L^2(T),$$

is a well-defined linear continuous operator. trf will stand for the trace of F, provided F has a summable trace (cf. Corollary 2.2). Note that in this case trf is an element of $L^2(T^{p-1})$.

Theorem 3.3. *Let ξ be given by (3.2). Then $\int_T \xi * dW$ exists if and only if the operator* F *defined by (3.3) has a summable trace. In this case*

$$\int_T \xi * dW = I_{p+1}(f) + pI_{p-1}(\mathrm{trf}).$$

Proof. We have

$$a_n = \int_T \xi(t)\phi_n(t)\,dm(t)$$

$$= \int_T \left(\int_{T^p} f(s_1,\dots, s_p,t)\,dW(s_1)\dots dW(s_p)\right)\phi_n(t)\,dm(t)$$

$$= I_p(g_n),$$

where $g_n(s_1,\dots, s_p) = \int_T f(s_1,\dots, s_p,t)\phi_n(t)\,dm(t)$, and the interchange of the multiple Itô-Wiener and usual integration can be easily verified for simple functions and extended to the general case by the usual approximation argument. By Itô's recurrence formula (cf. [5], Thm. 2.2) we get

$$(3.4) \qquad a_n \int_T \phi_n \, dW = I_p(g_n) I_1(\phi_n)$$

$$= I_{p+1}(g_n \otimes \phi_n) + p I_{p-1}(h_n),$$

where $(g_n \otimes \phi_n)(t_1, \ldots, t_{p+1}) = g_n(t_1, \ldots, t_p)\phi_n(t_{p+1}) =$ $\langle f(t_1, \ldots, t_p, \cdot), \phi_n(\cdot)\rangle \phi_n(t_{p+1})$ and

$$h_n(s_1, \ldots, s_{p-1}) = \int_{T^2} \phi_n(s) f(s_1, \ldots, s_{p-1}, s, t)\phi_n(t)\, dm(s)\, dm(t).$$

We observe now that $\Sigma_n g_n \otimes \phi_n$ converges to f in $L^2(T^{p+1})$ and consequently $\Sigma_n I_{p+1}(g_n \otimes \phi_n)$ converges to $I_{p+1}(f)$ in $L^2(\Omega)$. Therefore, in view of (3.4), $\int \xi * dW$ exists if and only if $\Sigma_n h_n$ converges in $L^2(T^{p-1})$ for every CONS $\{\phi_n\} \subset L^2(T)$, which means that F has a summable trace. Since $\Sigma_n h_n = \mathrm{tr} f$, (3.4) completes the theorem. □

Example 3.4.

$$\int_0^1 H_n(B(t),t) * dB(t) = \frac{1}{n+1} H_{n+1}(B(1),1) + \frac{n}{2} \int_0^1 H_{n-1}(B(t),t)\, dt,$$

where $H_n(x,t)$ is the Hermite polynomial of degree n defined by

$$H_n(x,t) = (-t)^n e^{x^2/2t} \frac{\partial^n}{\partial x^n} e^{-x^2/2t}, \quad t > 0.$$

Proof. Indeed, $H_n(B(t),t) = I_n(f(\cdot,t))$, where $f(s_1,\ldots, s_n,t) = 1_{[0,t]^n}(s_1,\ldots, s_n)$. Therefore the symmetrization of f in the last two variables is given by

$$\tilde{f}(s_1,\ldots, s_{n-1},s,t) = \begin{cases} \frac{1}{2} & \text{if } \max\{s_1,\ldots, s_{n-1}\} \le \max\{s,t\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\phi_n\}$ be a CONS in $L^2[0,1]$. We have

$$\sum_n \int_0^1\int_0^1 \phi_n(s)f(s_1,\ldots, s_{n-1},s,t)\phi_n(t)dsdt$$

$$= \sum_n \int_0^1\int_0^1 \phi_n(s)\tilde{f}(s_1,\ldots, s_{n-1},s,t)\phi_n(t)dsdt$$

$$= 2^{-1}\sum_n \int_0^1\int_0^1 \phi_n(s)1[\max\{s_1,\ldots, s_{n-1}\} \le \max\{s,t\}]\phi_n(t)dsdt$$

$$= 2^{-1}\sum_n \int_0^1\int_0^1 \phi_n(s)(1 - 1[\max\{s_1,\ldots, s_{n-1}\} > s]1[\max\{s_1,\ldots, s_{n-1}\}>t])\phi_n(t)dsdt$$

$$= 2^{-1}\sum_n (\langle 1_{[0,1]},\phi_n\rangle^2 - \langle 1_{[0,\max\{s_1,\ldots, s_{n-1}\}]},\phi_n\rangle^2)$$

$$= 2^{-1}(1 - \max\{s_1,\ldots, s_{n-1}\})$$

in $L^2([0,1]^{n-1})$. Hence F has a summable trace and $(\mathrm{tr}f)(s_1,\ldots, s_{n-1}) = 2^{-1}(1 - \max\{s_1,\ldots, s_{n-1}\})$. By Theorem 3.3 $\int_0^1 H_n(B(t),t)*dB(t)$ exists. To evaluate this integral we observe that

$$I_{n+1}(f) = I_{n+1}(1_{[0,s_{n+1}]^n}(s_1,\ldots, s_n)) = (n + 1)^{-1}I_{n+1}(1_{[0,1]^{n+1}})$$

$$= (n + 1)^{-1}H_{n+1}(B(1),1),$$

and, since a multiple Itô-Wiener integral can be expressed by usual Itô integral (cf. [5], Theorem 5.1),

$$nI_{n-1}(\mathrm{trf}) = 2^{-1}nI_{n-1}(1 - \max\{s_1,\ldots, s_{n-1}\})$$

$$= 2^{-1}n! \int_0^1 \int_0^{s_{n-1}} \cdots \int_0^{s_2} (1 - s_{n-1})dB(s_1)\ldots dB(s_{n-2})dB(s_{n-1})$$

$$= 2^{-1}n(n - 1)\int_0^1 (1 - s_{n-1})H_{n-2}(B(s_{n-1}),s_{n-1})dB(s_{n-1})$$

$$= 2^{-1}n\int_0^1 (1 - s)dX(s),$$

where $X(s) = H_{n-1}(B(s),s) = (n - 1)\int_0^s H_{n-2}(B(u),u)dB(u)$. Integrating by parts (note that sample paths of X are continuous) we get

$$\int_0^1 (1 - s)dX(s) = (1 - s)X(s)\Big|_0^1 + \int_0^1 X(s)ds,$$

which yields $nI_{n-1}(\mathrm{trf}) = 2^{-1}n\int_0^1 X(s)ds$, and completes the example.

<u>Example 3.5</u>. $\int_0^1 B(1 - t)*dB(t)$ does not exist.

<u>Proof</u>. Indeed, $B(1 - t) = \int_0^1 f(s,t)dB(s)$, where f is a symmetric function defined on $[0,1]^2$ by

$$f(s,t) = \begin{cases} 1 & \text{if } s + t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.3 and Proposition 2.1 it is sufficient to show that $F = \tilde{F}$ is not a nuclear operator on $L^2[0,1]$, where

$$(F\phi)(t) = \int_0^1 f(s,t)\phi(s)ds = \int_0^{1-t} \phi(s)ds.$$

Consider the sequences $\{\phi_n\}$ and $\{\psi_n\}$ of orthonormal functions in $L^2[0,1]$:

$$\phi_n(t) = \sqrt{2}\cos(2\pi nt), \qquad n \geq 1,\ t \in [0,1],$$

and

$$\psi_n(t) = \sqrt{2}\sin(2\pi n(1-t)).$$

Then

$$\sum_n \langle F\phi_n, \psi_n \rangle = 2\sum_n \int_0^1 \left(\int_0^{1-t} \cos(2\pi ns)ds \right) \sin 2\pi n(1-t)dt$$

$$= \sum_n \frac{1}{2\pi n} = \infty,$$

which shows that F is not a nuclear operator.

4. Integration of general Wiener L^2-functionals

Throughout this section we shall assume that

$$\xi \in L^2(T \times \Omega, A \otimes F^W, m \otimes P).$$

According to the well-known Itô-Wiener theorem which says that

$$L^2(\Omega, F^W, P) = \sum_{p=0}^{\infty} \oplus K_p,$$

where K_p is the p-th homogeneous chaos, we may decompose $\xi(t)$ into an orthogonal series

$$\xi(t) = \sum_{p=0}^{\infty} \xi_p(t),$$

where $\xi_p = \{\xi_p(t): t \in T\} \subset K_p$ and $\xi_0(t) = E\xi(t)$. Since we can always choose ξ_p as measurable processes belonging to $L^2(T \times \Omega)$ we also have

$$\xi = \sum_{p=0}^{\infty} \xi_p \quad \text{in } L^2(T \times \Omega). \tag{4.1}$$

Moreover, each ξ_p, $p \geq 1$ can be represented by a multiple Itô-Wiener integral

$$\xi_p(t) = I_p(f_p(\cdot, t)), \; t \in T \tag{4.2}$$

where $f_p = f_p(s_1, \ldots, s_p, t) \in L^2(T^{p+1})$ is symmetric in $s_1, \ldots, s_p$ for each fixed t. We set $f_0(t) = E\xi(t)$ and as usual $I_0(c) = c$. Further $\tilde{f}_p$ will denote the symmetrization of f_p in the last two variables. For every $p \geq 1$ we define an operator

$$F_p : L^2(T) \to L^2(T; L^2(T^{p-1}))$$

by

$$(F_p\phi)(t) = \int_T \phi(s) f_p(\cdot,s,t)\,dm(s), \quad \phi \in L^2(T).$$

By Proposition 2.1, F_p has a summable trace if and only if for every $h \in L^2(T^{p-1})$ $\tilde{F}_{p,h} : L^2(T) \to L^2(T)$ is a nuclear operator, where

$$(\tilde{F}_{p,h}\phi)(t) = \int_T \phi(s) \langle \tilde{f}_p(\cdot,s,t), h(\cdot)\rangle\,dm(s), \quad \phi \in L^2(T).$$

We define

$$(4.3) \qquad [\![F_p]\!] = \sup\{\|\tilde{F}_{p,h}\|_T : \|h\| \le 1,\ h \in L^2(T^{p-1})\},$$

where $\|A\|_T$ denotes the nuclear norm of an operator A (cf. e.g. [2], p. 111). Note that (4.3) always makes sense, whether or not F_p has a summable trace. Clearly, if $[\![F_p]\!] < \infty$, then F_p has a summable trace. The converse is also true and this simply follows by the Closed Graph Theorem applied to the linear mapping $h \to \tilde{F}_{p,h}$. Finally, $\mathrm{tr} f_p$ will stand for the trace of F_p, provided $[\![F_p]\!] < \infty$.

Theorem 4.1. *Assume that*

$$\Lambda^2(\{f_p\}) := \|f_0\|^2_{L^2(T)} + \sum_{p=1}^{\infty} (p+1)!\{\|f_p\|^2_{L^2(T^{p+1})} + [\![F_p]\!]^2\}$$

is finite. Then $\int_T \xi^* dW$ *exists,*

$$(4.4) \qquad \int_T \xi^* dW = I_1(f_0) + \sum_{p=1}^{\infty} [I_{p+1}(f_p) + pI_{p-1}(\mathrm{tr} f_p)]$$

in $L^2(\Omega)$ *and*

$$\left\| \int_T \xi^* dW \right\|_{L^2(\Omega)} \leq \sqrt{2} A(\{f_p\}).$$

Proof. We have

$$(4.5) \qquad \sum_{n=1}^{N} \int_T \xi \phi_n dm \int_T \phi_n dW = \sum_{p=0}^{\infty} S_{p,N},$$

where

$$S_{p,N} = \sum_{n=1}^{N} \int_T \xi_p \phi_n dm \int_T \phi_n dW$$

and the series $\Sigma_{p=0}^{\infty} S_{p,N}$ converges in $L^1(\Omega)$ for each $N \geq 1$.

By Theorem 3.3 for every $p \geq 1$,

$$(4.6) \qquad S_{p,N} \to I_{p+1}(f_p) + pI_{p-1}(\mathrm{tr} f_p) \quad \text{in } L^2(\Omega) \text{ as } N \to \infty,$$

and obviously $S_{0,N} \to I_1(f_0)$. Using (3.4) we have

$$(4.7) \qquad S_{p,N} = I_{p+1}(f_{p,N}) + pI_{p-1}(k_{p,N}),$$

where $\qquad f_{p,N}(t_1, \ldots, t_{p+1}) = \sum_{n=1}^{N} \langle f_p(t_1, \ldots, t_p, \cdot), \phi_n(\cdot) \rangle \phi_n(t_{p+1})$

and $\qquad k_{p,N}(s_1, \ldots, s_{p-1}) = \sum_{n=1}^{N} \int_{T^2} \phi_n(s) \tilde{f}_p(s_1, \ldots, s_{p-1}, s, t) \phi_n(t) dm(s) dm(t).$

Hence $\|f_{p,N}\| \leq \|f_p\|$ and

$$\|k_{p,N}\| = \sup\{\langle k_{p,N},h\rangle : \|h\| \leq 1,\ h \in L^2(T^{p-1})\}$$

$$\leq \sup\{\|\tilde{F}_{p,h}\|_T : \|h\| \leq 1,\ h \in L^2(T^{p-1})\} = [\![F_p]\!].$$

Therefore, for every $r > q \geq 1$ and $N \geq 1$

$$\|\sum_{p=q}^{r} S_{p,N}\|^2 = \|\sum_{p=q}^{r} [I_{p+1}(f_{p,N}) + pI_{p-1}(k_{p,N})]\|^2$$

$$\leq 2\|\sum_{p=q}^{r} I_{p+1}(f_{p,N})\|^2 + 2\|\sum_{p=q}^{r} pI_{p-1}(k_{p,N})\|^2$$

$$\leq 2\sum_{p=q}^{r} (p+1)!\|f_{p,N}\|^2 + 2\sum_{p=q}^{r} pp!\|k_{p,N}\|^2$$

$$\leq 2\sum_{p=q}^{r} (p+1)![\|f_p\|^2 + [\![F_p]\!]^2],$$

which shows that $\|\Sigma_{p=q}^{r} S_{p,N}\|_{L^2(\Omega)} \to 0$ uniformly in N as $q,r \to \infty$. Combining this (4.5) and (4.6) Theorem 4.1 follows. □

<u>Proposition 4.2</u>. *If* $\xi = \Sigma_{p=0}^{q}\xi_p$, *where* $q < \infty$, *then* $A(\{f_p\}) < \infty$ *is also a necessary condition for the existence of* $\int\xi * dW$.

<u>Proof</u>. Let Q_p be the orthogonal projection of $L^2(\Omega,F^W,P)$ onto K_p. Using (4.5) and (4.7) we get for each $0 \leq p < q$

$$Q_p(\int_T \xi * dW) = \lim_{N\to\infty} Q_p(\sum_{n=1}^{N} \int_T \xi\phi_n dm \int_T \phi_n dW)$$

$$= \lim_{N\to\infty} \sum_{p=0}^{q} Q_p(S_{p,N})$$

$$= \lim_{N\to\infty}[I_p(f_{p-1,N}) + (p+1)I_p(k_{p+1,N})]$$

$$= I_p(f_{p-1}) + (1 + p) \lim_{N\to\infty} I_p(k_{p+1,N}).$$

Therefore $\{k_{p+1,N}\}_{N=1}^{\infty}$ converges in $L^2(T^p)$ for any orthonormal basis $\{\phi_n\} \subset L^2(T)$. This implies that F_{p+1} has a summable trace and $[\![F_{p+1}]\!] < \infty$. □

We do not know whether or not $A(\{f_p\}) < \infty$ is necessary for the existence of $\int \xi^* dW$ in the general case. Nevertheless Theorem 4.1 gives a straightforward way to establish the integrability of ξ. Clearly the basic difficulty is in getting an upper bound for $[\![F_p]\!]$. We now use certain Sobolev-space type conditions on the f_p's to upper bound $[\![F_p]\!]$.

Theorem 4.3. *Let* $T = [0,1]$ *and* m *be Lebesgue measure on* T. *For* $p \ge 1$ *and* $\alpha > 0$ *we define*

$$\mathcal{U}_\alpha^2(f_p) := \|f_p\|^2_{L^2(T^{p+1})} + \int_{T^2} |u - v|^{-1-2\alpha} \|\tilde{f}_p(\cdot,u) - \tilde{f}_p(\cdot,v)\|^2_{L^2(T^p)} du dv.$$

Assume that for some $\alpha > \frac{1}{2}$,

$$\mathcal{U}_\alpha^2(\{f_p\}) := \|f_0\|^2_{L^2(T)} + \sum_{p=1}^{\infty} (p+1)!\,\mathcal{U}_\alpha^2(f_p)$$

is finite. Then $\int_0^1 \xi^* dB$ *exists, (4.4) holds and*

$$\left\|\int_0^1 \xi^* dB\right\|_{L^2(\Omega)} \le C\,\mathcal{U}_\alpha(\{f_p\}),$$

where C *depends only on* α.

Proof. Since $\|\tilde{f}_p\| \le \|f_p\|$ and $\mathcal{U}_\alpha^2(f_p) < \infty$, the function $[0,1] \ni t \to \tilde{f}_p(\cdot,t) \in L^2([0,1]^p)$ has absolutely convergent Fourier series, i.e.

$$\tilde{f}_p(s_1,\dots, s_p,t) = \sum_{n\in\mathbb{Z}} c_{p,n}(s_1,\dots, s_p)e^{2\pi int}$$

in $L^2([0,1]^{p+1})$, where $\Sigma_{n\in\mathbb{Z}}\|c_{p,n}\|_{L^2([0,1]^p)} < \infty$; c.f. [9], proof of Theorem 2. Moreover

$$\sum_{n\in\mathbb{Z}}\|c_{p,n}\| \le C\mathcal{U}_\alpha(f_p),$$

where C depends only on α. Put $\chi_n(t) = e^{2\pi int}$.

Let $h \in L^2([0,1]^{p-1})$ and let $\{\phi_n\}$ and $\{\psi_n\}$ be two sequences of orthonormal functions in $L^2[0,1]$. We have

$$\begin{aligned}\sum_j|<\tilde{F}_{p,h}\phi_j,\psi_j>| &\le \sum_n \sum_j|\int_0^1<c_{p,n}(\cdot,s),h(\cdot)>\phi_j(s)ds|\,|\int_0^1\chi_n(t)\psi_j(t)dt|\\ &\le \sum_n(\sum_j|\int_0^1<c_{p,n}(\cdot,s),h(\cdot)>\phi_j(s)ds|^2)^{\frac{1}{2}}(\sum_j|\int_0^1\chi_n(t)\psi_j(t)dt|^2)^{\frac{1}{2}}\\ &\le \sum_n(\int_0^1|<c_{p,n}(\cdot,s),h(\cdot)>|^2ds)^{\frac{1}{2}}\\ &\le \sum_n\|c_{p,n}\|\,\|h\| \le C\mathcal{U}_\alpha(f_p)\|h\|.\end{aligned}$$

Hence $\|\tilde{F}_{p,h}\|_T \le C\mathcal{U}_\alpha(f_p)\|h\|$ which yields $[\![F_p]\!] \le C\mathcal{U}_\alpha(f_p)$. Therefore $A^2(\{f_p\}) \le (C^2 + 1)\mathcal{U}_\alpha^2(\{f_p\})$ and Theorem 4.1 completes the proof. □

A sufficient condition for the integrability of ξ, stronger than that of Theorem 4.3, can be written in terms of the covariance functions of the component processes $\{\xi_p\}$.

Theorem 4.4. *Let* T = [0,1] *and* m *be Lebesgue measure on* T. *If for some* $\alpha > 1/2$,

$$N_\alpha^2(\xi) := \|\xi_0\|_{L^2(T)} + \sum_{p=1}^{\infty} p\{\|\xi_p\|^2_{L^2(T\times\Omega)} + \int_{T^2} \frac{E|\xi_p(u) - \xi_p(v)|^2}{|u-v|^{1+2\alpha}}\, dudv\}$$

is finite, then $\int_0^1 \xi^* dB$ *exists, (4.4) holds and*

$$\|\int_0^1 \xi^* dB\| \le CN_\alpha(\xi),$$

where C *depends only on* α.

Proof. Since

$$\|\xi_p\|^2_{L^2(T\times\Omega)} + \int_{T^2} \frac{E|\xi_p(u) - \xi_p(v)|^2}{|u-v|^{1+2\alpha}}$$

$$= p!\{\|\tilde f_p\|^2_{L^2(T^{p+1})} + \int_{T^2} |u-v|^{-1-2\alpha} \|f_p(\cdot,u) - f_p(\cdot,v)\|^2_{L^2(T^p)}\, dudv\}$$

and

$$|\langle \tilde F_{p,h}\phi,\psi\rangle| \le 2^{-1}(|\langle F_{p,h}\phi,\psi\rangle| + |\langle F_{p,h}\psi,\phi\rangle|),$$

where $F_{p,h}$ is defined similarly to $\tilde F_{p,h}$ with $\tilde f_p$ replaced by f_p, the inequality $A(\{f_p\}) \le \text{Const } N_\alpha(\xi)$ follows by the same arguments as those used in the proof of Theorem 4.3.

5. Evaluation of the integral by the Fisk-Stratonovich procedure

Throughout this section (T,m) will be the unit interval with Lebesgue measure.

Let $\xi(t)$, $t \in [0,1]$ be a stochastic process. We say that a (generalized) Fisk-Stratonovich integral of ξ exists if

$$(5.1) \qquad S_\pi := \sum_{j=1}^{n} 2^{-1}[\xi(t_{j-1}) + \xi(t_j)][B(t_j) - B(t_{j-1})]$$

converges in probability as mesh $(\pi) \to 0$, where π runs over all finite partitions $0 = t_0 < t_1 <\ldots< t_n = 1$ $(n \in \mathbb{N})$ of $[0,1]$, and we write

$$\int_0^1 \xi(t)\circ dB(t) = \lim_{\mathrm{mesh}(\pi)\to 0} S_\pi .$$

Note that we do not require in this definition any kind of measurability of ξ.

In this section we shall study the relationship between $\int \xi * dB$ and $\int \xi \circ dB$. Let ξ be given by (4.1) and (4.2). Put $D_+ = \{(s,t)\colon 0 \le s < t \le 1\}$ and $D_- = \{(s,t)\colon 0 \le t < s \le 1\}$. Define f_p^+ (f_p^-, respectively) as the restriction of the function

$$[0,1]^2 \ni (s,t) \to f_p(\cdot,s,t) \in L^2([0,1]^{p-1})$$

to D_+ (D_-, respectively).

Proposition 5.1. *Let* $\xi = \Sigma_{p=0}^{q} \xi_p$, $q < \infty$, *be a mean-square continuous stochastic process. Assume that for every* $1 \le p \le q$ *the functions* f_p^+ *and* f_p^- *are continuous and possess the extensions (also denoted by* f_p^+ *and* f_p^-, *respectively) to continuous functions from* $\bar{D}_+$ ($\bar{D}_-$, *respectively) into* $L^2([0,1]^{p-1})$. *Then* $\int_0^1 \xi(t)\circ dB(t)$ *exists and*

$$\int_0^1 \xi(t)\circ dB(t) = I_1(f_0) + \sum_{p=1}^{q} [I_{p+1}(f_p) + pI_{p-1}(g_p)],$$

where

$$g_p(\cdot) = 2^{-1}\int_0^1 [f_p^-(\cdot,s,s) + f_p^+(\cdot,s,s)]ds.$$

Proof. Clearly we may assume that $\xi = \xi_p$, where $0 \le p \le q$. The case $p = 0$ is obvious. Let $p \ge 1$ and let $\pi = \{t_0, \ldots, t_n\}$ be a partition of $[0,1]$. Using Itô's recurrence formula we get

$$S_\pi = \sum_{j=1}^{n} 2^{-1}[\xi_p(t_{j-1}) + \xi_p(t_j)][B(t_j) - B(t_{j-1})]$$

(5.2)
$$= \sum_{j=1}^{n} I_p(2^{-1}[f_p(\cdot,t_{j-1}) + f_p(\cdot,t_j)])I_1(1_{[t_{j-1},t_j]})$$

$$= I_{p+1}(f_{p,\pi}) + pI_{p-1}(g_{p,\pi})$$

where

$$f_{p,\pi}(\cdot,t) = \sum_{j=1}^{n} 2^{-1}[f_p(\cdot,t_{j-1}) + f_p(\cdot,t_j)]1_{[t_{j-1},t_j]}(t)$$

and

$$g_{p,\pi}(\cdot) = 2^{-1}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} [f_p^-(\cdot,s,t_{j-1}) + f_p^+(\cdot,s,t_j)]ds.$$

Since the mapping $[0,1]^2 \ni (s,t) \to f_p(\cdot,s,t) \in L^2([0,1]^{p-1})$ is continuous and uniformly bounded on $D_+ \cup D_-$, the mapping $[0,1] \ni t \to f_p(\cdot,t) \in L^2([0,1]^p)$ is continuous. Hence $f_{p,\pi} \to f_p$ in $L^2([0,1]^{p+1})$ as mesh $(\pi) \to 0$. By the continuity of f_p^+ and f_p^- on $\overline{D}_+$ and $\overline{D}_-$, respectively, $g_{p,\pi} \to g_p$ in $L^2([0,1]^{p-1})$ as mesh $(\pi) \to 0$.

Proposition 5.2. [illegible]ose that ξ *satisfies the assumptions of Proposition* [illegible] $\int_0^1 \xi(t)*dB(t)$ [illegible]. *Then*

$$\int_0^1 \xi(t)*dB(t) = \int_0^1 \xi(t)\circ dB(t).$$

Proof. We have for a.e. $(s,t) \in [0,1]^2$

$$(5.3) \qquad \tilde{f}_p(\cdot,s,t) = \frac{1}{2}[f_p^+(\cdot,s\wedge t,s\vee t) + f_p^-(\cdot,s\vee t,s\wedge t)],$$

and the function on the right-hand side of (5.3) is continuous in $(s,t) \in [0,1]^2$. By Proposition 4.2 for every $h \in L^2([0,1]^{p-1})$ $\tilde{F}_{p,h}$ is a nuclear operator on $L^2[0,1]$. Hence for every $h \in L^2([0,1]^{p-1})$

$$\begin{aligned} \langle trf_p, h\rangle &= \text{Trace } (\tilde{F}_{p,h}) \\ &= \int_0^1 2^{-1}\langle f_p^+(\cdot,s,s) + f_p^-(\cdot,s,s), h(\cdot)\rangle ds \\ &= \langle g_p, h\rangle \end{aligned}$$

(cf. e.g. Theorem 3.4.4 [2]). Proposition 5.1 and Theorem 4.1 complete the proof. □

Example 5.3. In Example 3.4 we showed that $\int_0^1 H_n(B(t),t)*dB(t)$ exists and we evaluated the integral. Since in this case the assumptions of Proposition 5.1 are satisfied, Proposition 5.2 provides an alternative way of evaluation of $\int_0^1 H_n(B(t),t)*dB(t)$. Using Proposition 5.2 and Theorem 1.1, Chap. III in [4] we get

$$\begin{aligned} \int_0^1 H_n(B(t),t)*dB(t) &= \int_0^1 H_n(B(t),t)\circ dB(t) \\ &= \int_0^1 H_n(B(t),t)dB(t) + \frac{1}{2}\int_0^1 d\langle H_n(B(s),s),B(s)\rangle \\ &= \frac{1}{n+1} H_{n+1}(B(1),1) + \frac{n}{2}\int_0^1 H_{n-1}(B(s),s)ds. \end{aligned}$$

Remark 5.4. The assumptions of Proposition 5.1 do not guarantee the existence of $\int\xi{*}dB$. Indeed, it is well-known that there exists a continuous symmetric kernel $k:[0,1]^2 \to \mathbb{R}$ such that the corresponding integral operator is not nuclear (cf. e.g. [2], p. 124). Put $\xi(t) = \int_0^t k(s,t)dB(s)$. Then the assumptions of Proposition 5.1 are satisfied, but by Theorem 3.3 $\int\xi{*}dB$ does not exist.

Below are given simple examples of Gaussian processes for which the Fisk-Stratonovich integral exists while the series expansion (1.1) fails to converge.

Example 5.5. $\int_0^1 B(1-t)\circ dB(t) = B^2(\frac{1}{2}) + 2\int_{\frac{1}{2}}^1 B(1-t)dB(t)$, but $\int_0^1 B(1-t){*}dB(t)$ does not exist (cf. Example 3.5).

Proof. Indeed, it is easy to check that

$$\sum_{j=1}^{j_0} 2^{-1}(B(1-t_{j-1}) + B(1-t_j)][B(t_j) - B(t_{j-1})] \to B^2(\tfrac{1}{2}) + \int_{\frac{1}{2}}^1 B(1-t)dB(t),$$

and

$$\sum_{j=j_0+1}^{n} 2^{-1}[B(1-t_{j-1}) + B(1-t_j)][B(t_j) - B(t_{j-1})] \to \int_{\frac{1}{2}}^1 B(1-t)dB(t)$$

in $L^2(\Omega)$ as $\mathrm{mesh}(\pi) \to 0$, where $\pi = \{t_0,\ldots, t_n\}$ is a partition of $[0,1]$ and $j_0 = \max\{j: t_j < \frac{1}{2}\}$.

Example 5.6. Let $\xi(t)$, $t \in [0,1]$ be a non-anticipating stochastic process given by

$$\xi(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2} \\ B(t-\frac{1}{2}) & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

Then $\int_0^1 \xi(t)\circ dB(t) = \int_{\frac{1}{2}}^1 B(t-\frac{1}{2})dB(t)$, while $\int_0^1 \xi * dB$ does not exist.

Proof. Indeed, it is elementary to check that $\Sigma_{j=1}^n (\xi(t_j) - \xi(t_{j-1}))(B(t_j) - B(t_{j-1})) \to 0$ in $L^2(\Omega)$ as $\text{mesh}(\pi) \to 0$, which implies that the Fisk-Stratonovich integral exists and is equal to the ordinary Itô integral of ξ.

On the other hand $\xi(t) = \int_0^1 f(s,t)dB(s)$, where $f(s,t) = 1$ if $0 \le s \le \frac{1}{2}$ and $s + \frac{1}{2} \le t \le 1$ and $= 0$ otherwise. Hence $\tilde{f}$ is given by

$$\tilde{f}(s,t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le s \le \frac{1}{2} \text{ and } s + \frac{1}{2} \le t \le 1 \\ \frac{1}{2} & \text{if } \frac{1}{2} \le s \le 1 \text{ and } 0 \le t \le s - \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Consider the sequences $\{\phi_n\}$ and $\{\psi_n\}$ of orthonormal vectors in $L^2[0,1]$ given by

$$\phi_n(t) = \cos 4\pi nt, \quad 0 \le t \le 1$$

and

$$\psi_n(t) = \begin{cases} \sin 4\pi n(t + \frac{1}{2}) & \text{if } 0 \le t \le \frac{1}{2} \\ 0 & \text{otherwise .} \end{cases}$$

Then

$$\sum_n \langle \tilde{F}\phi_n, \psi_n \rangle = \frac{1}{2}\sum_n \int_0^{\frac{1}{2}} \left(\int_{t+\frac{1}{2}}^1 \phi_n(s)ds\right)\psi_n(t)dt$$

$$= \sum_n \frac{-1}{32\pi n} = -\infty,$$

which completes the example.

Intuitively speaking, the existence of the Fisk-Stratonovich integral requires some kind of continuity of the process ξ while conditions for $\int_0^1 \xi * dB$ are of a different nature. Hence it is also easy to give an example of a process ξ for which, conversely, $\int \xi * dB$ exists but $\int \xi \circ dB$ does not exist.

Example 5.7. Let A be a dense Borel subset of [0,1] such that $0 < m(A) < 1$. Put $\psi(t) = 1_A(t)$, $f(s,t) = \psi(s)\psi(t)$ and $\xi(t) = \int_0^1 f(s,t)dB(s) = (\int_0^1 \psi dB)\psi(t)$. Clearly $\tilde{F}\phi = F\phi = \langle\psi\ \phi\rangle\psi$ is nuclear, while $S_\pi = 0$ if $\pi = \{t_0,\ldots, t_n\} \subset [0,1]\setminus A$ and $S_\pi = (\int_0^1 \psi dB)B(1)$ if $\pi \subset A$, where S_π is defined by (5.1). Therefore $\int \xi \circ dB$ does not exist.

Proposition 5.2 shows the equality of both integrals $\int \xi \circ dB$ and $\int \xi * dB$ under certain additional assumptions. This is an open question if the existence of both integrals suffices for their equality.

Examples 5.5 and 5.6 indicate that the existence of the series expansion (1.1) is a quite strong property of the process ξ. Below are given certain Sobolev-space type conditions, similar to those proposed by Kuo and Rusek [6], which imply the existence and equality of both integrals, $\int \xi \circ dB$ and $\int \xi * dB$. Since the proof in [6] seems to contain some gaps and the final condition differs from ours in the value of a coefficient ($p!$ instead of $(p+1)!$), we present a complete proof of this result. Moreover our proof does not use the theory of Sobolev spaces, which makes it more elementary.

In what follows below

$$f_p^+(s_1,\ldots,s_p,t) = f_p(s_1,\ldots,s_{p-1},s_p \wedge t, s_p \vee t)$$

and

$$f_p^-(s_1,\ldots,s_p,t) = f_p(s_1,\ldots,s_{p-1},s_p \vee t, s_p \wedge t).$$

<u>Theorem 5.8</u>. *Let ξ be given by (4.1) and (4.2), where f_0 is continuous. Assume that for some $\alpha > \frac{1}{2}$,*

$$M_\alpha^2(\{f_p\}) = \|f_0\|^2_{L^2(T)} + \sum_{p=1}^{\infty} (p+1)!\,[U_\alpha^2(f_p^-) + U_\alpha^2(f_p^+)]$$

is finite, where $U_\alpha^2(\cdot)$ is defined in Theorem 4.3. Then both integrals $\int_0^1 \xi \circ dB$ and $\int_0^1 \xi * dB$ exist, they are equal a.s. and (4.4) holds. Moreover,

$$\left\|\int_0^1 \xi * dB\right\|_{L^2(\Omega)} \le CM_\alpha(\{f_p\}),$$

where C depends only on α.

<u>Proof</u>. Since $f_p^- + f_p^+ = 2\tilde{f}_p$ and $f_p^- 1_{T^{p-1}\times D_-} + f_p^+ 1_{T^{p-1}\times D_+} = f_p$, we get $U_\alpha^2(f_p) \le \frac{1}{2}[U_\alpha^2(f_p^+) + U_\alpha^2(f_p^-)]$. In view of Theorem 4.3 $\int \xi * dB$ exists. Using the same argument as in the proof of Theorem 4.3 we have

$$f_p^-(\cdot, t) = \sum_{n\in\mathbb{Z}} c_{p,n}^-(\cdot)\chi_n(t) \tag{5.4}$$

in $L^2([0,1]^{p+1})$, where $\chi_n(t) = \exp(i2\pi nt)$ and $\Sigma_n \|c_{p,n}^-\|_{L^2(T^p)} \le CU_\alpha(f_p^-)$. A similar expansion we have for f_p^+ with $c_{p,n}^-$ replaced by $c_{p,n}^+$ in (5.4).

Let $\pi = \{t_0, \ldots, t_k\}$ be a partition of $[0,1]$. We have

$$S_\pi = \sum_{p=0}^{\infty} S_{p,\pi} \quad \text{in } L^2(\Omega),$$

where $S_{p,\pi}$ is defined by (5.1) with ξ replaced by ξ_p. Using (5.2) and (5.4) we get

$$S_{p,\pi} = I_{p+1}(f_{p,\pi}) + pI_{p-1}(g_{p,\pi}),$$

where

$$f_{p,\pi}(\cdot,t) = \sum_{n\in\mathbb{Z}} c^-_{p,n}(\cdot) \sum_{j=1}^{k} 2^{-1}[1_{T^{p-1}\times D_-}(\cdot,t_{j-1})\chi_n(t_{j-1}) + 1_{T^{p-1}\times D_-}(\cdot,t_j)\chi_n(t_j)]1_{(t_{j-1},t_j]}(t)$$

$$+ \sum_{n\in\mathbb{Z}} c^+_{p,n}(\cdot) \sum_{j=1}^{k} 2^{-1}[1_{T^{p-1}\times D_+}(\cdot,t_{j-1})\chi_n(t_{j-1}) + 1_{T^{p-1}\times D_+}(\cdot,t_j)\chi_n(t_j)]1_{(t_{j-1},t_j]}(t)$$

$$= \sum_{n\in\mathbb{Z}} c^-_{p,n}(\cdot)\psi^-_{n,\pi}(\cdot,t) + \sum_{n\in\mathbb{Z}} c^+_{p,n}(\cdot)\psi^+_{n,\pi}(\cdot,t),$$

and

$$g_{p,\pi}(s_1,\dots,s_{p-1}) = 2^{-1}\sum_{n\in\mathbb{Z}} \int_0^1 c^-_{p,n}(s_1,\dots,s_{p-1},s)\chi^-_{n,\pi}(s)ds$$

$$+ 2^{-1}\sum_{n\in\mathbb{Z}} \int_0 c^+_{p,n}(s_1,\dots,s_{p-1},s)\chi^+_{n,\pi}(s)ds.$$

Here $\chi^-_{n,\pi}(t) = \chi_n(t_{j-1})$ and $\chi^+_{n,\pi}(t) = \chi_n(t_j)$ if $t \in (t_{j-1},t_j]$, $j = 1,\dots, k$. Since $|\psi^-_{n,\pi}| \le 1$ and $|\psi^+_{n,\pi}| \le 1$ we obtain

$$\|f_{p,\pi}\|_{L^2(T^{p+1})} \le \sum_n(\|c^-_{p,n}\|_{L^2(T^p)} + \|c^+_{p,n}\|_{L^2(T^p)}) \le C[\mathcal{U}_\alpha(f^+_p) + \mathcal{U}_\alpha(f^-_p)]$$

and by Schwartz inequality

$$\|g_{p,\pi}\|_{L^2(T^{p-1})} \le 2^{-1}\sum_{n\in\mathbb{Z}} \|\int_T c^-_{p,n}(\cdot,s)\chi^-_{n,\pi}(s)ds\|_{L^2(T^{p-1})}$$

$$+ 2^{-1}\sum_{n\in\mathbb{Z}} \|\int_T c^+_{p,n}(\cdot,s)\chi^+_{n,\pi}(s)ds\|_{L^2(T^{p-1})}$$

$$\le 2^{-1}\sum_{n\in\mathbb{Z}} (\|c^-_{p,n}\|_{L^2(T^p)} + \|c^+_{p,n}\|_{L^2(T^p)})$$

$$\le C2^{-1}[\mathcal{U}_\alpha(f_p^+) + \mathcal{U}_\alpha(f_p^-)].$$

Therefore

$$\|\sum_{p=q}^{r} S_{p,\pi}\|^2 \le 2\|\sum_{p=q}^{r} I_{p+1}(f_{p,\pi})\|^2 + 2\|\sum_{p=q}^{r} pI_{p-1}(g_{p,\pi})\|^2$$

$$\le 2\sum_{p=q}^{r}(p+1)![\|f_{p,\pi}\|^2 + \|g_{p,\pi}\|^2]$$

$$\le 3C^2\sum_{p=q}^{r}(p+1)![\mathcal{U}_\alpha^2(f_p^+) + \mathcal{U}_\alpha^2(f_p^-)] \to 0$$

as $p,q \to \infty$, uniformly in all finite partitions π of $[0,1]$.

To complete the proof it is enough to show that for each $p \ge 1$, $S_{p,\pi} \to I_{p+1}(f_p) + pI_{p-1}(\mathrm{tr} f_p)$ in $L^2(\Omega)$ as mesh $(\pi) \to 0$. To this end we shall show that $\|f_{p,\pi} - f_p\| \to 0$ and $\|g_{p,\pi} - \mathrm{tr} f_p\| \to 0$ as mesh $(\pi) \to 0$.

Using (5.4) we have

$$\|f_{p,\pi} - f_p\|_{L^2(T^{p+1})} \le \sum_{n\in\mathbb{Z}} \|c_{p,n}^-(\psi_{n,\pi}^- - 1_{T^{p-1}\times D_-}\chi_n)\|_{L^2(T^{p+1})}$$

$$+ \sum_{n\in\mathbb{Z}} \|c_{p,n}^+(\psi_{n,\pi}^+ - 1_{T^{p-1}\times D_+}\chi_n)\|_{L^2(T^{p+1})} \to 0$$

as mesh $(\pi) \to 0$ by the Dominated Convergence Theorem.

Since

$$\mathrm{tr} f_p = \mathrm{tr}\tilde{f}_p = 2^{-1}\mathrm{tr} f_p^- + 2^{-1}\mathrm{tr} f_p^+$$

and both f_p^- and f_p^+ are symmetric in the last two variables we obtain

$$\mathrm{tr} f_p^- = \sum_{n\in\mathbb{Z}} \int_{T^2} \chi_n(s) f_p^-(\cdot,s,t)\overline{\chi}_n(t)\,ds\,dt$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 c_{p,n}^-(\cdot,s)\chi_n(s)\,ds,$$

where $\bar{a}$ denotes the complex conjugate to a. A similar expression we obtain for trf_p^+.

Finally

$$\|g_{p,\pi} - trf_p\|_{L^2(T^{p-1})} \le 2^{-1} \sum_{n\in\mathbb{Z}} \left\|\int_0^1 c_{p,n}^-(\cdot,s)(\chi_{n,\pi}^-(s) - \chi_n(s))ds\right\|_{L^2(T^{p-1})}$$

$$+ 2^{-1} \sum_{n\in\mathbb{Z}} \left\|\int_0^1 c_{p,n}^+(\cdot,s)(\chi_{n,\pi}^+(s) - \chi_n(s))ds\right\|_{L^2(T^{p-1})}$$

$$\le 2^{-1} \sum_{n\in\mathbb{Z}} \| c_{p,n}^-\|_{L^2(T^p)} \|\chi_{n,\pi}^- - \chi_n\|_{L^2(T)}$$

$$+ 2^{-1} \sum_{n\in\mathbb{Z}} \|c_{p,n}^+\|_{L^2(T^p)} \|\chi_{n,\pi}^+ - \chi_n\|_{L^2(T)} \to 0$$

as mesh $(\pi) \to 0$ by the Dominated Convergence Theorem. The proof of Theorem 5.8 is complete. □

It occurs that a simple condition $N_\alpha(\xi) < \infty$, for some $\alpha > \frac{1}{2}$, given in Theorem 4.4 implies not only the existence of $\int \xi^* dB$ but also the integrability of ξ in the Stieltjes sense.

Theorem 5.9. *Assume that for some* $\alpha > \frac{1}{2}$, $N_\alpha(\xi) < \infty$, *where* $N_\alpha(\xi)$ *is defined in Theorem 4.4. Then for every partition* $\pi = \{t_0,\dots, t_k\}$ *of* $[0,1]$ *and any choice* $t_j^* \in [t_{j-1},t_j]$, $j = 1,\dots, k$,

$$S_\pi^* = \sum_{j=1}^{k} \xi(t_j^*)[B(t_j) - B(t_{j-1})]$$

converges to $\int_0^1 \xi^* dB$ *as* mesh$(\pi) \to 0$.

Proof. By (4.8), $t \to f_p(\cdot,t)$ has absolutely convergent Fourier series, i.e. f_p can be presented in the form similar to (5.4). Starting from this representation and following essentially all the steps in the proof of Theorem 5.8 we complete the proof of Theorem 5.9. □

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